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We propose a method to produce entangled states of several particles starting from a Bose-Einstein condensate. In the proposal, a single fast $\pi/2$ pulse is applied to the atoms and due to the collisional interaction, the subsequent free time evolution creates an entangled state involving all atoms in the condensate. The created entangled state is a spin-squeezed state which could be used to improve the sensitivity of atomic clocks.

The possibility of creating and manipulating entangled states of many–particle systems has recently boosted the field of quantum information since it may yield new applications which rely on the basic principles of Quantum Mechanics. These applications are particularly important in the fields of computation, communication, and atomic clocks and frequency standards[1,2]. So far, up to four atoms have been entangled in a controlled way[3,4]. In fact, these experiments may already lead to a reduction of the so–called projection noise in atomic clocks by a factor of two[5]. Scaling up those systems to obtain higher degrees of noise reduction is very difficult in general, since one needs to produce quantum pure states and manipulate them in the absence of decoherence. On the other hand, the recent experimental achievement of Bose–Einstein condensation[6–8] has raised a lot of attention since it may lead to some applications in several fields of Science. Some of these applications are based on the fact that condensates can be considered as pure states at the single particle level. Thus, a natural question is to investigate whether Bose–Einstein condensates can also be used in some applications of quantum information. In this paper we show that this is indeed the case. In fact, we demonstrate that with present technology one can obtain atomic entangled states, with such a degree of entanglement that one could reduce the projection noise in atomic clocks by several orders of magnitude.

Let us consider a set of N two–level atoms (or qubits) confined by some external trap. In order to describe the internal properties of these atoms, it is convenient to consider them as spin 1/2 particles and to use total angular momentum operators, $\vec{J} = \sum_{n=1}^{N} \vec{j}^{(n)}$, where $\vec{j}^{(n)}$ is the angular momentum operator corresponding to the n–th atom. The entanglement properties of the atoms can be expressed in terms of the variances and expectation values of these operators. In the appendix we show that if

$$\xi^2 \equiv \frac{N(\Delta J_{\vec{n}_1})^2}{\langle J_{\vec{n}_2}\rangle^2 + \langle J_{\vec{n}_3}\rangle^2} < 1, \qquad (1)$$

where $J_{\vec{n}} \equiv \vec{n}\cdot\vec{J}$ and the $\vec{n}$'s are mutually orthogonal unit vectors, then the state of the atoms is non–separable (i.e. entangled). The parameter ξ thus characterizes the atomic entanglement. In fact, it has been shown that in the context of atomic clocks, where projection noise is currently the main source of noise[9], atoms in a state with $\xi^2 < 1$ can reduce the frequency noise (variance in the frequency measurements) or the measuring time to obtain a desired precision by a factor ξ^2 as compared to the case in which one uses atoms in an uncorrelated state[2]. In this context, one uses the term "spin squeezed states" to denote states with $\xi^2 < 1$. Several theoretical proposals for noise reduction have been made[10,11], and a weak squeezing of the spin has recently been produced experimentally[12]. In the following we will show how one can reduce ξ^2 by several orders of magnitude using Bose–Einstein condensates, exploiting the collisional interactions between the atoms.

We propose to generate spin–squeezed states starting from a two component weakly interacting Bose-Einstein condensate. This experimental set–up is present in several laboratories[13,14]. We will denote by $|a\rangle$ and $|b\rangle$ the two relevant internal atomic states, and we will assume that the interactions do not change the internal state (below we show how this can be achieved in practice). This situation is described by the second quantized Hamiltonian

$$\begin{aligned} H = \sum_{j=a,b} & \int d^3r \hat{\Psi}_j^\dagger(\mathbf{r}) H_{0,j} \hat{\Psi}_j(\mathbf{r}) \\ & + \frac{1}{2}\sum_{j=a,b} U_{jj} \int d^3r \hat{\Psi}_j^\dagger(\mathbf{r})\hat{\Psi}_j^\dagger(\mathbf{r})\hat{\Psi}_j(\mathbf{r})\hat{\Psi}_j(\mathbf{r}) \\ & + U_{ab} \int d^3r \hat{\Psi}_a^\dagger(\mathbf{r})\hat{\Psi}_b^\dagger(\mathbf{r})\hat{\Psi}_a(\mathbf{r})\hat{\Psi}_b(\mathbf{r}), \end{aligned} \qquad (2)$$

where $H_{0,j}$ is the one particle Hamiltonian for atoms in state j including the kinetic energy and the external trapping potential $V_j(\mathbf{r})$, $\hat{\Psi} j(\mathbf{r})$ is the field operator for atoms in the state j, $U_{jk} = 4\pi\hbar^2 a_{jk}/m$ is the strength of the interaction between particles of type j and k, parameterized by the scattering length a_{jk}, and m is the atomic mass.

Let us assume that we start with a Bose–Einstein condensate in state $|a\rangle$ at very low temperature ($T \simeq 0$), so that all the atoms are in a single particle (motional) state $|\phi_0\rangle$. A fast $\pi/2$ pulse between the states $|a\rangle$ and $|b\rangle$ prepares the atoms in the state $|\phi_0\rangle^{\otimes N} \otimes (|a\rangle + |b\rangle)^{\otimes N}/2^{N/2}$ which is an eigenstate of the J_x operator with eigenvalue $N/2$. If we choose in (1) $\vec{n}_1 = [0, \cos(\theta), \sin(\theta)]$ and $\vec{n}_2$ along the x axis, we have $\xi_\theta^2 = 1$ at $t = 0$. We first show that after certain time ξ_θ^2 is reduced for some value of θ. From the Hamiltonian (2) we find the time derivative of

ξ_θ^2 at $t=0$

$$\frac{d}{dt}\xi_\theta^2 = \sin(2\theta)\frac{(N-1)(U_{aa}+U_{bb}-2U_{ab})}{2\hbar}\int d^3r|\phi_0|^4. \tag{3}$$

This equation immediately shows that spin squeezing will be produced for certain angles θ if $U_{aa}+U_{bb}\neq 2U_{ab}$. This condition is fulfilled in the Na experiments[14], and could also be achieved by appropriate tuning of the scattering lengths in the Rb experiments[13].

Having seen that spin squeezing can be produced with Bose–Einstein condensates, let us quantify the amount of squeezing which may be obtained. For the sake of simplicity we will assume identical trapping potentials $V_a(\mathbf{r}) = V_b(\mathbf{r})$ and identical coupling constants for interaction between atoms in the same internal state $a_{aa} = a_{bb} > a_{ab}$. Physically, this could correspond to the $|F=1, M_F=\pm 1\rangle$ hyperfine states of Na trapped in an optical dipole trap. Due to the symmetry of these states their scattering lengths and trapping potentials will be identical; moreover, due to angular momentum conservation there are no spin exchanging collisions between these states as required by our model. This is exactly the experimental setup used in Ref. 14. In this experiment it is shown that these states have an anti-ferromagnetic interaction $a_{ab} < a_{aa}$ which according to (3) enables the production of squeezed states. Actually, one has to slightly modify such an experimental set–up in order to avoid spin changing collisions that populate the state $|M_F = 0\rangle$[15]. One way of avoiding this is to couple the $F=1$ manifold to the $F=2$ with a far off-resonant blue detuned π-polarized microwave field. The Clebsch-Gordan coefficient for the $|F=1, M_F=0\rangle \rightarrow |F=2, M_F=0\rangle$ is larger than the coefficients for the $|F=1, M_F=\pm 1\rangle \rightarrow |F=2, M_F=\pm 1\rangle$ transitions so that the $|F=1, M_F=0\rangle$ state is raised in energy with respect to the $|F=1, M_F=\pm 1\rangle$ states, and spin exchanging collisions become energetically forbidden.

The assumption $a_{aa} = a_{bb}$ has several advantages. Firstly, it reduces the effect of fluctuations in the total particle number. If $a_{aa}\neq a_{bb}$ the mean spin will perform a N dependent rotation around the z-axis. Fluctuations in the number of particles will introduce an uncertainty in the direction of the spin which effectively reduces the average value and introduces noise into the system. With $a_{aa} = a_{bb}$ the mean spin remains in the x-direction independent of the number of atoms in the trap. Secondly, this condition ensures a large spatial overlap of different components of the wavefunction. After the $\pi/2$ pulse the spatial wavefunction is no longer in the equilibrium state and it will start oscillating. Furthermore, since the state of the system is now distributed over a range of number of particles in the $|a\rangle$ state (N_a), and since this number enters into the time evolution, the N_a dependent wavefunctions ϕ_a and ϕ_b are different for particles in the states $|a\rangle$ and the $|b\rangle$. With $a_{aa} = a_{bb}$, ϕ_a and ϕ_b are identical if N_a equals the average number $N/2$. In the limit of large N, the width of the distribution on different N_a's is much smaller than N_a and all the spatial wavefunctions are approximately identical $\phi_a(N_a,t)\approx\phi_b(N_a,t)\approx\phi_0(t)$. This approximation is only valid if $a_{aa} > a_{ab}$ where small deviations from the average wavefunction perform small oscillations. In the opposite case the deviations grow exponentially[16] resulting in a reduction of the overlap of the a and the b wavefunctions and a reduced squeezing. The advantages mentioned above could also be achieved with $a_{aa}\neq a_{bb}$ by using the breath–together solutions proposed in Ref. 16, but here we only consider $a_{aa} = a_{bb}$.

Before analyzing quantitatively the complete system, let us estimate the amount of spin squeezing we can reach with our proposal by using a simple model. Assuming the same wavefunction ϕ_0 for both $|a\rangle$ and $|b\rangle$ atoms are constant and independent of the number N_a, the spin dependent part of the Hamiltonian (2) may be written as $H_{\rm spin} = \hbar\chi J_z^2$, where χ depends on the scattering lengths and the wavefunction ϕ_0. The spin squeezing produced by this Hamiltonian can be calculated exactly[17]. In the limit of large N, the minimum obtainable squeezing parameter is $\xi_\theta^2 = \frac{1}{2}\left(\frac{3}{N}\right)^{2/3}$, which indicates that our proposal might produce a reduction of ξ^2 by a factor of $\sim N^{2/3}$ which would be more than three orders of magnitude with 10^5 atoms in the condensate.

To justify that our proposal is actually capable of producing squeezing resembling the results of the Hamiltonian $H_{\rm spin}$, we have performed a direct numerical integration of the equations describing the system. Following the procedure developed in Ref. 16 we expand the state of the system at $t=0$ on the Fock states $|N_a : \phi_a(N_a,t); (N-N_a) : \phi_b(N_a,t)\rangle$ which contain N_a ($N-N_a$) particles in the $|a\rangle$ ($|b\rangle$) state with spatial wavefunction ϕ_a (ϕ_b). Using the time dependent coupled Gross-Pitaevski equations we propagate the three–dimensional spatial wavefunctions $\phi_a(N_a,t)$ and $\phi_b(N_a,t)$ in time, and by including the phase factor $A(N_a,t)$ derived in 16 we find the state of the system at any later time. For numerical convenience we have assumed a spherically symmetric potential $V(r) = m\omega^2r^2/2$. Together with the prediction from the simple Hamiltonian $H_{\rm spin}$ the result of the simulation is shown in Fig. 1. The parameter χ is chosen such that the reduction of J_x obtained from the solution in Ref. 17 is consistent with the results of Ref. 16. The two curves are roughly in agreement confirming that the system is able to approximate the results of the Hamiltonian $H_{\rm spin}$. The numerical solution shows fluctuations due to the oscillations of the spatial wavefunction. The large dips at $\omega t\approx 4$, 9, 13, and 18 are the points where the spatial wavefunctions reach the initial width. At these instants the overlap of the wavefunctions is maximal and the two curves are very close (up to a factor of two). The small dips at $\omega t =$2, 7, 11, and 16 corresponds to the points of maximum compression. With the realistic parameters used in the figure, our simulation suggests that three orders of magnitude squeezing is possible. Also, note the time scale in the figure. The maximally squeezed state is reached after approximately two oscillation periods in the trap. For a fixed ratio of the scattering lengths a_{ab}/a_{aa}, the optimal time scales as $(a_{aa}/d_0)^{-2/5}N^{-1/15}$, where $d_0 = \sqrt{\hbar/(m\omega)}$ is the width of the ground state of the harmonic potential.

The analysis so far has left out a number of possible imperfections. Specifically, we have assumed that all scattering length are real so that no atoms are lost from the trap and we have not considered the role of thermal particles. To estimate the effect particle losses we have performed a Monte Carlo simulation[18] of the evolution of squeezing from the Hamiltonian H_{spin}. The particle loss is phenomenologically taken into account by introducing a loss rate Γ which is identical for atoms in the $|a\rangle$ and the $|b\rangle$ state. In Fig. 2 we shown the obtainable squeezing in the presence of loss. Approximately 10 % of the atoms are lost at the time $\chi t \approx 6 * 10^{-4}$ where the squeezing is maximally without loss. With the parameters of Fig. 1 this time corresponds to roughly two trapping periods. Such a large loss is an exaggeration of the loss compared to current experiments and the simulation indicate that even under these conditions, squeezing of nearly two orders of magnitude may be obtained. On the other hand, the effects of thermal particles can be suppressed at sufficiently low temperatures but due to the robustness with respect to particle loses shown in Fig. 2, we expect to obtain high squeezing even at some finite temperatures.

In conclusion we believe that we have presented a simple and robust method to produce entangled states of a large number of atoms with present technology. The produced entangled state are interesting in fundamental physics and the inclusion of our procedure in atomic clocks could be a practical application of quantum information and Bose-Einstein condensates. In future experiments with negligible particle loss even for very long interaction times, the Hamiltonian H_{spin} could also be used to produce a maximally entangled state of any number of atoms[3,19].

Appendix In this appendix we prove Eq. (1). An N-particle density matrix ρ is defined to be separable (non-entangled) if it can be decomposed into

$$\rho = \sum_k P_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes ... \otimes \rho_k^{(N)}, \tag{4}$$

where the coefficients P_k are positive real numbers fulfilling $\sum_k P_k = 1$, and $\rho_k^{(i)}$ is a density matrix for the i'th particle. The variance of J_z may be described as $(\Delta J_z)^2 = \frac{N}{4} - \sum_k P_k \sum_i \langle j_z^{(i)} \rangle_k^2 + \sum_k P_k \langle J_z \rangle_k^2 - \langle J_z \rangle^2$, and using Cauchy-Schwarz's inequality and $\langle j_x^{(i)} \rangle_k^2 + \langle j_y^{(i)} \rangle_k^2 + \langle j_z^{(i)} \rangle_k^2 \leq 1/4$ we find three inequalities for separable states $\sum_k P_k \langle J_z \rangle_k^2 \geq \langle J_z \rangle^2$, $-\sum_k P_k \sum_i \langle j_z^{(i)} \rangle_k^2 \geq -\frac{N}{4} + \sum_k P_k \sum_i \langle j_x^{(i)} \rangle_k^2 + \langle j_y^{(i)} \rangle_k^2$, and $\langle J_x \rangle^2 \leq N \sum_k P_k \sum_i \langle j_x^{(i)} \rangle_k^2$. From these inequalities we immediately find that any separable state obeys $\xi^2 > 1$ and hence any state with $\xi^2 < 1$ is non-separable.

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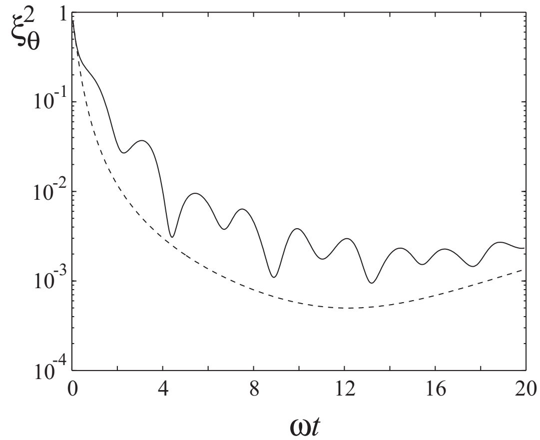

FIG. 1. Reduction in the squeezing parameter ξ^2. A fast $\pi/2$ pulse between two internal states is applied to all atoms in the condensate. The subsequent free time evolution results in a strong squeezing of the total spin. The angle θ is chosen such that ξ_θ^2 is minimal. The solid line is the results of a direct numerical integration of the time dependent Gross-Pitaevski equations with parameters $a_{aa}/d_0 = 6*10^{-3}$, $a_{bb} = 2a_{ab} = a_{aa}$, and $N = 10^5$. The dashed curve shows the squeezing obtained from the Hamiltonian $H_{\rm spin}$.

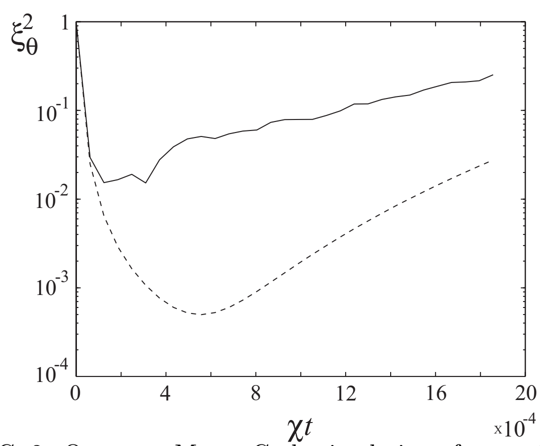

FIG. 2. Quantum Monte Carlo simulation of squeezing in the presence of loss. The full line shows squeezing obtained from the Hamiltonian $H_{\rm spin}$ with particle loss described by a constant loss rate $\Gamma = 200\chi$. The dashed curve shows squeezing without particle loss.